

## 4.4 - Linear Independence

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**Definition:** If  $S = \{v_1, v_2, \dots, v_r\}$  is a set of two or more vectors in a vector space  $V$ , then  $S$  is said to be a **linearly independent set** if no vector in  $S$  can be expressed as a linear combination of the others. A set that is not linearly independent is said to be **linearly dependent**. If  $S$  has only one vector, we will agree that it is linearly independent if and only if that vector is nonzero.

$\{\vec{e}_1, \vec{e}_2\}$  is lin. indep in  $\mathbb{R}^2$ .

$\{\vec{e}_1, \vec{e}_2, (2, 5)\}$  is not. Because

$$(2, 5) = 2\vec{e}_1 + 5\vec{e}_2$$

**Theorem 4.4.1** A nonempty set  $S = \{v_1, v_2, \dots, v_r\}$  in a vector space  $V$  is linearly independent if and only if the only coefficients satisfying the vector equation  $k_1 v_1 + k_2 v_2 + \dots + k_r v_r = \vec{0}$  are  $k_1 = 0, k_2 = 0, \dots, k_r = 0$ .

pf: ( $\Rightarrow$ )  $S = \{\vec{v}_1, \dots, \vec{v}_r\}$  is lin. indep.

Suppose  $\exists k_1, k_2, \dots, k_r$  not all zero  $\exists k_1 \vec{v}_1 + k_2 \vec{v}_2 + \dots + k_r \vec{v}_r = \vec{0}$ . Let  $k_i \neq 0$

$$k_i \vec{v}_i = -k_1 \vec{v}_1 - \dots - k_r \vec{v}_r$$

$$\Rightarrow \vec{v}_i = -\frac{k_1}{k_i} \vec{v}_1 - \dots - \frac{k_r}{k_i} \vec{v}_r$$

this contradicts linear independence.

( $\Leftarrow$ )  $k_1 \vec{v}_1 + \dots + k_r \vec{v}_r = \vec{0}$  only if  $k_i = 0 \forall i$ .

Suppose  $\{\vec{v}_i\}$  is not linearly independent.

$$\text{Then } \vec{v}_i = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_r \vec{v}_r$$

$$\Rightarrow \vec{v}_i - c_1 \vec{v}_1 - c_2 \vec{v}_2 - \dots - c_r \vec{v}_r = \vec{0},$$

contradicting that all the coefficients are zero.

#3 In each part, determine whether the vectors are linearly independent or are linearly dependent in  $\mathbb{R}^4$ .

a.  $(3, 8, 7, \vec{v}_1)$ ,  $(1, 5, 3, -1)$ ,  $(2, -1, 2, 6)$ ,  $(4, 2, 6, 4)$

b.  $(3, 0, -3, 6)$ ,  $(0, 2, 3, 1)$ ,  $(0, -2, -2, 0)$ ,  $(-2, 1, 2, 1)$

a. Can we have  $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4 = \vec{0}$

if  $a, b, c, d$  are not all zero?

$$\left[ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \mid \vec{0} \right]$$

iff  $a=b=c=d=0$ , we have the trivial solution.

The system  $A\vec{x} = \vec{0}$  has only the trivial solution iff

$A$  is invertible  $\Leftrightarrow \det(A) \neq 0$

$\Leftrightarrow \text{rref}(A) = I_n$

$$\begin{vmatrix} 3 & 1 & 2 & 4 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 8 & 8 \\ 8 & 5 & -1 & 2 \\ 7 & 3 & 2 & 6 \\ -3 & -1 & 6 & 4 \end{vmatrix} \stackrel{\text{it turns out}}{=} 0$$

The system has nontrivial solutions.

The set is linearly dependent.

(We can find  $-\vec{v}_1 + \vec{v}_2 - \vec{v}_3 + \vec{v}_4 = \vec{0}$ .)

b) We would find  $\det(A) \neq 0$

The set is independent.

#6 Determine all values of  $k$  for which the following matrices are linearly independent in  $M_{22}$ .

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

We want  $k$  such that if  $a \begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix} + b \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix} + c \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , then  $a=b=c=0$ .

$$\begin{aligned} a - b + 2c &= 0 \\ a + bk + c &= 0 \\ ak + b + 3c &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has only the trivial solution if  $\det(A) \neq 0$

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{vmatrix} = 2k^2 + 1 - 3$$
$$2k + 2 - 2k^2 + 2 = 0$$
$$3k - k + 2$$

$$\Rightarrow (k-2)(k+1) = 0 \Rightarrow k = -1, 2$$

The set is independent for  $k \neq -1, 2$

#9

- a. Show that the three vectors  $\mathbf{v}_1 = (0, 3, 1, -1)$ ,  $\mathbf{v}_2 = (6, 0, 5, 1)$ ,  $\mathbf{v}_3 = (4, -7, 1, 3)$  form a linearly dependent set in  $\mathbb{R}^4$ .
- b. Express each vector in part (a) as a linear combination of the other two.

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$$

$$\begin{bmatrix} 0 & 6 & 4 \\ 3 & 0 & -7 \\ 1 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -7/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_i: a = \frac{7}{3}c, \quad b = -\frac{2}{3}c$$

$$\text{Let } c = 3, \text{ then } a = 7, \quad b = -2$$

$$\text{So } 7\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3 = \vec{0}.$$

solving for  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  satisfies (b).

#14 In each part, let  $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be multiplication by  $A$  and let  $\mathbf{u}_1 = (1, 0, 0)$ ,  $\mathbf{u}_2 = (2, -1, 1)$ , and  $\mathbf{u}_3 = (0, 1, 1)$ . Determine whether the set  $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$  is linearly independent in  $\mathbb{R}^3$ .

a.  $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

b.  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

$$\text{Step 1: } T_A(\vec{u}_1) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$T_A(\vec{u}_2) = A\vec{u}_2, \quad T_A(\vec{u}_3) = A\vec{u}_3$$

Then proceed as before.

**Theorem 4.4.2**

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{0} = \vec{0}$$

- a) A set with finitely many vectors that contains  $\vec{0}$  is linearly dependent.
- b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

**Theorem 4.4.3** Let  $S = \{v_1, v_2, \dots, v_r\}$  be a set of vectors in  $R^n$ . If  $r > n$ , then  $S$  is linearly dependent.

Consider  $(2, 3, 4), (-1, 5, 2), (4, 7, 3), (8, 1, 8)$

This yields :

$$\begin{bmatrix} 2 & -1 & 4 & 8 \\ 3 & 5 & 7 & 1 \\ 4 & 2 & 3 & 8 \end{bmatrix}$$

This leads to free variables.

**Definition:** If  $f_1 = f_1(x)$ ,  $f_2 = f_2(x)$ , ...,  $f_n = f_n(x)$  are functions that are  $n-1$  times differentiable on the interval  $(-\infty, \infty)$ , then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of  $f_1, f_2, \dots, f_n$ .

**Theorem 4.4.4** If the functions  $f_1, f_2, \dots, f_n$  have  $n-1$  continuous derivatives on the interval  $(-\infty, \infty)$ , and if the Wronskian of these functions is not identically zero on  $(-\infty, \infty)$ , then these functions form a linearly independent set of vectors in  $C^{(n-1)}(-\infty, \infty)$ .

**#19** Use the Wronskian to show that the following sets of vectors are linearly independent.

a.  $1, x, e^x$

b.  $1, x, x^2$

$$W(1, x, e^x) = \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x \neq 0$$

The set is independent.

Consider  $\{1, \cos 2x, \sin^2 x\}$

Note:  $\cos 2x = 1 - 2\sin^2 x$

The set is dependent.

$$W(1, \cos 2x, \sin^2 x) = \begin{vmatrix} 1 & \cos 2x & \sin^2 x \\ 0 & -2\sin 2x & \cancel{2\sin x \cos x}^{\sin 2x} \\ 0 & -4\cos 2x & 2\cos 2x \end{vmatrix}$$

$$= 1 \begin{vmatrix} -2\sin 2x & \sin 2x \\ -4\cos 2x & 2\cos 2x \end{vmatrix}$$

$$= -4\sin 2x \cos 2x + 4\sin 2x \cos 2x = 0$$